Rough set theory and concept lattices to solve fuzzy relation equations

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IJCRS 2024
May 19th
Cádiz, Spain – Halifax, Canada

Outline

M·CIS

Introduction

Rough sets

Modal-style operators

Multi-adjoint Relation Equations

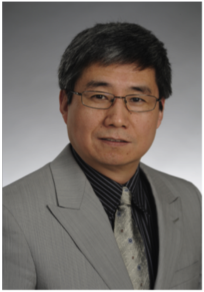
Reducing MARE

Conclusions and future work

Historical introduction

J. Medina.
Towards multi-adjoint property-oriented concept lattices.
Lecture Notes in Artificial Intelligence, 6401:159–166, 2010.

I met at RSKT-2010

Mathematics for Computational Intelligence Systems (M·CIS)

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- J. Carlos Díaz-Moreno
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- Víctor López-Marchante
- José Antonio Torné
- Samuel Molina-Ruiz


UCLM

Mathematical tools studied by our research group M·CIS

M·CIS mathematical tools

- Fuzzy sets.
- Fuzzy logic. Fuzzy Logic Programming.
- Fuzzy Formal Concept Analysis.
- Fuzzy Rough Sets.
- Fuzzy Relation Equations.
- Tools for the extraction, manipulation and prediction of information in databases.
- Linguistic description of data and automatic generation of natural language.

Formal concept analysis

- FCA, introduced by Wille in the eighties, arise as a useful tool for qualitative data analysis, which has become an important and appealing research topic.
- FCA is a theory of data analysis which identifies conceptual structures among data sets. It has been applied to linguistic databases, library and information science, . . .
- Handling uncertainty, imprecise data or incomplete information has become an important research topic in the recent years.

Rough sets

- Originally proposed by Pawlak.
- It was extended by Düntsch and Gediga, and later complemented by Yao, in order to consider two different sets, the set of objects and the set of attributes:
  - *Property-oriented concept lattice*
  - *Object-oriented concept lattice*
- Both FCA and RS theory have been related in several papers. As a consequence, we can apply the results presented in a formal concept analysis framework to rough set theory.

M. J. Benítez, J. Medina, E. Ramírez, and D. Ślęzak.
Rough-set-driven approach for attribute reduction in fuzzy formal concept analysis.
Fuzzy Sets and Systems, 391:117–138, 2020.

Awarded

PP-RAI Contest for the Most Influential Article on Rough Sets co-authored by Polish Researchers in 2020-2022.

Rough set theory

Information table/system

$(X, \mathcal{A})$, where X and $\mathcal{A}$ are finite, non-empty sets of objects and attributes, respectively. Each a in $\mathcal{A}$ corresponds to a mapping $\bar{a}\colon X \to V_a$, where V_a is the value set of a over X.

Example

	Temperature	Headache
x_1	Hight	Yes
x_2	Normal	Yes
x_3	Hight	Yes
x_4	Normal	No

B-indiscernibility relation

For every subset B of $\mathcal{A}$, the B-indiscernibility relation R_B is

$$R_B = \{(x, y) \in X \times X \mid \text{ for all } a \in B,\ a(x) = a(y)\}$$

which is an equivalence relation and the equivalence classes are denoted by $[x]_{R_B}$.

Example

$R_{\mathcal{A}}$	x_1	x_2	x_3	x_4
x_1	1	0	1	0
x_2	0	1	0	0
x_3	1	0	1	0
x_4	0	0	0	1

Rough set: lower and upper approximation

Example

- $[x_1]_{\mathcal{A}} = \{x_1, x_3\} = [x_3]_{\mathcal{A}}$
- $[x_2]_{\mathcal{A}} = \{x_2\}$
- $[x_4]_{\mathcal{A}} = \{x_4\}$

These classes are used to approximate sets.

Lower and upper approximation

Given $A \subseteq X$, its lower and upper approximation w.r.t. R_B are defined by

$$
\begin{aligned}
R_B{\downarrow}A &= \{x \in X \mid [x]_{R_B} \subseteq A\} \\
R_B{\uparrow}A &= \{x \in X \mid [x]_{R_B} \cap A \neq \varnothing\}
\end{aligned}
$$

Lower and upper approximation: Example

- $[x_1]_{\mathcal{A}} = \{x_1, x_3\} = [x_3]_{\mathcal{A}}$
- $[x_2]_{\mathcal{A}} = \{x_2\}, \quad [x_4]_{\mathcal{A}} = \{x_4\}$

$$R_B{\downarrow}A = \{x \in X \mid [x]_{R_B} \subseteq A\}$$

Example

If $A_1 = \{x_2, x_3\}$, then $\begin{cases} R_B{\downarrow}A_1 = \{x_2\} \\ R_B{\uparrow}A_1 = \{x_1, x_2, x_3\} \end{cases}$

If $A_2 = \{x_1, x_3\}$, then $\begin{cases} R_B{\downarrow}A_1 = \{x_1, x_3\} \\ R_B{\uparrow}A_1 = \{x_1, x_3\} \end{cases}$

Lower and upper approximation: Example

- $[x_1]_{\mathcal{A}} = \{x_1, x_3\} = [x_3]_{\mathcal{A}}$
- $[x_2]_{\mathcal{A}} = \{x_2\}, \quad [x_4]_{\mathcal{A}} = \{x_4\}$

$$R_B{\uparrow}A = \{x \in X \mid [x]_{R_B} \cap A \neq \varnothing\}$$

Example

If $A_1 = \{x_2, x_3\}$, then $\begin{cases} R_B{\downarrow}A_1 = \{x_2\} \\ R_B{\uparrow}A_1 = \{x_1, x_2, x_3\} \end{cases}$

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Lower and upper approximation: Properties

Properties

- $R_B{\downarrow}A \subseteq A \subseteq R_B{\uparrow}A$
- $A \subseteq B \Rightarrow \begin{cases} R{\downarrow}A \subseteq R{\downarrow}B \\ R{\uparrow}A \subseteq R{\uparrow}B \end{cases}$
- $R_B{\downarrow}(R_B{\uparrow}A) \subseteq R_B{\uparrow}A, \qquad R_B{\downarrow}A \subseteq R_B{\uparrow}(R_B{\downarrow}A)$

Property-oriented concept lattices

$R_{\mathcal{A}}$	x_1	x_2	x_3	x_4
x_1	1	0	1	0
x_2	0	1	0	0
x_3	1	0	1	0
x_4	0	0	0	1

- oBjects$=\{x_1, x_2, x_3, x_4\}$
- **A**ttributes$=\{a_1, a_2, a_3, a_4\}$
- Relation $R\colon A \times B \to \{0, 1\}$

R could not be an equivalence relation. Hence, given $b \in B$, the meaning of $[b]_R$ could be $Rb = \{a \in A \mid R(a, b) = 1\}$, and $[a]_R$ may be $aR = \{b \in B \mid R(a, b) = 1\}$, for all $a \in A$.

Given $X \subseteq B$ subset of objects, $Y \subseteq A$ subset of attributes

$$R{\downarrow}Y = \{b \in B \mid Rb \subseteq Y\}$$
$$R{\uparrow}X = \{a \in A \mid aR \cap X \neq \varnothing\}$$

Property-oriented concept lattices

R	x_1	x_2	x_3	x_4
a_1	1	0	1	0
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Given $X \subseteq B$ subset of objects, $Y \subseteq A$ subset of attributes

$$R{\downarrow}Y = \{b \in B \mid Rb \subseteq Y\} = Y^N = Y^{\downarrow^N}$$
$$R{\uparrow}X = \{a \in A \mid aR \cap X \neq \varnothing\} = X^\pi = X^{\uparrow_\pi}$$

Property-oriented concept lattices

FCA: Many-valued context

	Temperature		Headache	
	Normal	Hight	No	Yes
x_1	0	1	0	1
x_2	1	0	0	1
x_3	0	1	0	1
x_4	1	0	1	0

Given $X \subseteq B$ subset of objects, $Y \subseteq A$ subset of attributes

$$\{b \in B \mid Rb \subseteq Y\} = Y^N = Y^{\downarrow^N}$$
$$\{a \in A \mid aR \cap X \neq \varnothing\} = X^\pi = X^{\uparrow_\pi}$$

Modal-style operators

Given A, B, and $R\colon A \times B \to \{0,1\}$, the mappings ${}^{\pi}\colon 2^B \to 2^A$, ${}^{N}\colon 2^B \to 2^A$, defined, for each $X \subseteq B$, as:

$$X^{\pi} = \{a \in A \mid \text{ there is } b \in X, \text{ such that } aRb\} \qquad (1)$$

$$X^{N} = \{a \in A \mid \text{ for all } b \in B, \text{ if } aRb, \text{ then } b \in X\} \qquad (2)$$

Analogously, we can define the mappings: ${}^{\pi}\colon 2^A \to 2^B$, ${}^{N}\colon 2^A \to 2^B$.

These operators are called *possibility* and *necessity operators*, respectively.

They are composed to form Galois connections and, hence, new concept lattices: *object-oriented concept lattice* and *property-oriented concept lattice*.

Attribute classification

J. Medina.
Relating attribute reduction in formal, object-oriented and property-oriented concept lattices.
Computers & Mathematics with Applications 64:1992–2002, 2012.

Theorem

Let (A, B, R) be a formal context, and $\mathcal{B}(A, B, R^c)$, $L_o(A, B, R)$ and $L_p(A, B, R)$. For each $a \in A$, the following equivalences are obtained:

1. $a \in I_{f^c}$ *iff* $a \in I_o$ *iff* $a \in I_p$
2. $a \in K_{f^c}$ *iff* $a \in K_o$ *iff* $a \in K_p$
3. $a \in C_{f^c}$ *iff* $a \in C_o$ *iff* $a \in C_p$

Attribute reduction

Theorem

Given a formal context (A, B, R) and $D \subseteq A$, the following are equivalents:

- *D is an attribute reduct of the formal concept lattice $\mathcal{B}(A, B, R^c)$*
- *D is an attribute reduct of the object-oriented concept lattice $L_o(A, B, R)$*
- *D is an attribute reduct of the corresponding property-oriented concept lattice $L_p(A, B, R)$.*

Fuzzy POCL and OOCL

J. Medina.
Multi-adjoint property-oriented and object-oriented concept lattices.
Information Sciences, 190:95–106, 2012.

Given a frame $(L_1, L_2, P, \&_1, \dots, \&_l)$ and context (A, B, R, σ), we consider ${}^{\uparrow_\pi} \colon L_2^B \to L_1^A$, ${}^{\downarrow^N} \colon L_2^A \to L_1^B$:

$$\begin{aligned} g^{\uparrow_\pi}(a) &= \sup\{R(a,b) \,\&_{\sigma(b)}\, g(b) \mid b \in B\} \\ f^{\downarrow^N}(b) &= \inf\{f(a) \nwarrow_{\sigma(b)} R(a,b) \mid a \in A\} \end{aligned}$$

These definitions are generalizations of the classical and fuzzy possibility and necessity operators by Yao, Düntsch, Gediga, Georgescu, Popescu, Lai, Dubois, Prade, etc.

Multi-adjoint property-oriented concept lattice

The pair $({}^{\uparrow_\pi}, {}^{\downarrow^N})$ is an adjunction (isotone Galois connection), that is ${}^{\uparrow_\pi}$ and ${}^{\downarrow^N}$ are order-preserving; and they satisfy that $f^{\downarrow^N \uparrow_\pi} \preceq_1 f$, for all $f \in L_1^A$, and that $g \preceq_2 g^{\uparrow_\pi \downarrow^N}$, for all $g \in L_2^B$.

The set of the concepts

$$\mathcal{M}_{\pi N} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow_\pi} = f, f^{\downarrow^N} = g\}$$

together with the ordering $\preceq$ defined by
$\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ iff $g_1 \preceq_2 g_2$ (or $f_1 \preceq_1 f_2$) forms a complete lattice, $(\mathcal{M}_{\pi N}, \preceq)$, which is called *multi-adjoint property-oriented concept lattice*.

Definition

Given three sets U, V, W, a multi-adjoint frame and a mapping $\sigma\colon V \to \{1,\dots,n\}$ which assigns an adjoint triple to each variable, a MARE is an expression of the form

$$R \odot_\sigma X = T$$

where $R \in P^{U\times V}$, $T \in L_1^{U\times W}$ and $X \in L_2^{V\times W}$ is unknown.

The operator $\odot_\sigma$ is a sup-composition operator. It is convenient to recall that a MARE is equivalent to a set of systems of the form

$$\begin{array}{ccccccc}
R(u_1, v_1) \,\&_{\sigma(v_1)}\, x_1 & \vee & \cdots & \vee & R(u_1, v_m) \,\&_{\sigma(v_m)}\, x_m & = & t_1 \\
\vdots & & \vdots & & \vdots & & \vdots \\
R(u_r, v_1) \,\&_{\sigma(v_1)}\, x_1 & \vee & \cdots & \vee & R(u_r, v_m) \,\&_{\sigma(v_m)}\, x_m & = & t_r
\end{array}$$

Example. Multi-adjoint frame

$$([0,1]_8, \leq, \&_1^*, \swarrow_*^1, \nwarrow_1^*, \&_2^*, \swarrow_*^2, \nwarrow_2^*)$$

$x \&_1^* y = \frac{\lceil 8x^2y \rceil}{8}$, for all $x, y \in [0,1]_8$, and its corresponding residuated implications

$$z \swarrow_*^1 y = \begin{cases} 1 & \text{if } y = 0 \\ \min\left\{\frac{\lfloor 8\sqrt{z/y} \rfloor}{8}, 1\right\} & \text{otherwise} \end{cases}$$

$$z \nwarrow_1^* x = \begin{cases} 1 & \text{if } x = 0 \\ \min\left\{\frac{\lfloor 8z/x^2 \rfloor}{8}, 1\right\} & \text{otherwise} \end{cases}$$

$x \&_2^* y = \frac{\lceil 8xy^2 \rceil}{8}$, for all $x, y \in [0,1]_8$, and its corresponding residuated implications.

Example. Particular FRE

Consider $U = \{u_1, u_2, u_3, u_4, u_5\}$, $V = \{v_1, v_2, v_3, v_4, v_5\}$, $W = \{w\}$ and the FRE with sup-$\&_\sigma$-composition

$$R \odot_\sigma X = T \tag{3}$$

where $\sigma\colon V \to \{1, 2\}$ assigns v_1, v_2, v_4 to the first adjoint triple and v_3, v_5 to the second one,

$$R = \begin{pmatrix} 0.75 & 0.5 & 0 & 0.5 & 0.5 \\ 0.5 & 0.25 & 0.25 & 0.75 & 1 \\ 0.75 & 0.5 & 0.125 & 0 & 0.375 \\ 0.75 & 0.5 & 0 & 0.5 & 0.5 \\ 0.75 & 0.5 & 0.125 & 0 & 0.5 \end{pmatrix}, \quad T = \begin{pmatrix} 0.25 \\ 0.5 \\ 0 \\ 0.25 \\ 0 \end{pmatrix}$$

and $X \in [0,1]_8^{V \times W}$ is unknown.

Concept lattice associated with a MARE

The study of MARE will be based on the fact that every MARE can be assigned to a context.

Context associated with a MARE

The *context associated with the MARE* $R \odot_\sigma X = T$ is the multi-adjoint context (U, V, R, σ).

Every multi-adjoint context determines a property-oriented concept lattice, by means of the mappings $(^{\uparrow_\pi}, ^{\downarrow^N})$, which define an isotone Galois connection.

$$\mathcal{M}_{\pi N}(U, V, R) = \left\{(g, f) \in L_2^V \times L_1^U \mid g = f^{\downarrow^N}, f = g^{\uparrow_\pi}\right\}$$

Solvability and maximum solution of a MARE

J. C. Díaz-Moreno and J. Medina.
Multi-adjoint relation equations: Definition, properties and solutions using concept lattices.
Information Sciences, 253:100–109, 2013.

Theorem 1 [Solvability and maximum solution of a MARE]

Let $R \odot_\sigma X = T$ be a FRE and $(\mathcal{M}_{\pi N}, \preceq_{\pi N})$ the concept lattice associated with it. Then, $R \odot_\sigma X = T$ is solvable if and only if $T_w \in \mathcal{I}(\mathcal{M}_{\pi N})$ for all $w \in W$. In that case, $\overline{X} \in L_2^{V \times W}$ defined as $\overline{X}(v, w) = T_w^{\downarrow^N}(v)$ is the maximum solution of the equation.

Several approximation methods will be obtained from this result, using the associated concept lattice.

Example. Particular FRE

Consider $U = \{u_1, u_2, u_3, u_4, u_5\}$, $V = \{v_1, v_2, v_3, v_4, v_5\}$, $W = \{w\}$ and the FRE with sup-$\&_\sigma$-composition

$$R \odot_\sigma X = T \tag{4}$$

where $\sigma\colon V \to \{1, 2\}$ assigns v_1, v_2, v_4 to the first adjoint triple and v_3, v_5 to the second one,

$$R = \begin{pmatrix} 0.75 & 0.5 & 0 & 0.5 & 0.5 \\ 0.5 & 0.25 & 0.25 & 0.75 & 1 \\ 0.75 & 0.5 & 0.125 & 0 & 0.375 \\ 0.75 & 0.5 & 0 & 0.5 & 0.5 \\ 0.75 & 0.5 & 0.125 & 0 & 0.5 \end{pmatrix}, \quad T = \begin{pmatrix} 0.25 \\ 0.5 \\ 0 \\ 0.25 \\ 0 \end{pmatrix}$$

and $X \in [0, 1]_8^{V \times W}$ is unknown.

Example. Checking solvability

We will check whether FRE (4) is solvable by the characterization theorem based on its associated context (U, V, R, σ).

Specifically, we will see if $T = T^{\downarrow^N \uparrow_\pi}$, since W is a singleton, that is, T only has one column. Indeed, making the corresponding computations, the following chain of equalities holds

$$T^{\downarrow^N \uparrow_\pi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.875 \\ 0 \end{pmatrix}^{\uparrow_\pi} = \begin{pmatrix} 0.25 \\ 0.5 \\ 0 \\ 0.25 \\ 0 \end{pmatrix} = T$$

Novel advances

Attribute reduction theory

D. Lobo, V. López-Marchante, and J. Medina.
Reducing fuzzy relation equations via concept lattices.
Fuzzy Sets and Systems, 463:108465, 2023.

Reducing MARE

Reducing MARE is a process in which redundant information is removed from its coefficient matrix.

Definition [Reduced MARE]

Let $Y \subseteq U$ and consider the fuzzy relations $R_Y = R_{|Y \times V}$, $T_Y = T_{|Y \times W}$. The equation $R_Y \odot_\sigma X = T_Y$ is called *Y-reduced equation of the MARE $R \odot_\sigma X = T$*.

Reducing a MARE in a consistent set preserves its solution set.

Theorem 2 [Reduction theorem]

Let $R \odot_\sigma X = T$ be a solvable MARE and Y a consistent set of it associated context (U, V, R, σ). The Y-reduced equation of $R \odot_\sigma X = T$ is solvable. Moreover, they have the same solution set.

Example. Multi-adjoint frame

$$([0,1]_8, \leq, \&_1^*, \swarrow^1_*, \nwarrow^*_1, \&_2^*, \swarrow^2_*, \nwarrow^*_2)$$

$x \&_1^* y = \frac{\lceil 8x^2y \rceil}{8}$, for all $x, y \in [0,1]_8$, and its corresponding residuated implications

$$z \swarrow^1_* y = \begin{cases} 1 & \text{if } y = 0 \\ \min\left\{\frac{\lfloor 8\sqrt{z/y} \rfloor}{8}, 1\right\} & \text{otherwise} \end{cases}$$

$$z \nwarrow^*_1 x = \begin{cases} 1 & \text{if } x = 0 \\ \min\left\{\frac{\lfloor 8z/x^2 \rfloor}{8}, 1\right\} & \text{otherwise} \end{cases}$$

$x \&_2^* y = \frac{\lceil 8xy^2 \rceil}{8}$, for all $x, y \in [0,1]_8$, and its corresponding residuated implications.

Example. Particular FRE

Consider $U = \{u_1, u_2, u_3, u_4, u_5\}$, $V = \{v_1, v_2, v_3, v_4, v_5\}$, $W = \{w\}$ and the FRE with sup-$\&_\sigma$-composition

$$R \odot_\sigma X = T \tag{5}$$

where $\sigma\colon V \to \{1, 2\}$ assigns v_1, v_2, v_4 to the first adjoint triple and v_3, v_5 to the second one,

$$R = \begin{pmatrix} 0.75 & 0.5 & 0 & 0.5 & 0.5 \\ 0.5 & 0.25 & 0.25 & 0.75 & 1 \\ 0.75 & 0.5 & 0.125 & 0 & 0.375 \\ 0.75 & 0.5 & 0 & 0.5 & 0.5 \\ 0.75 & 0.5 & 0.125 & 0 & 0.5 \end{pmatrix}, \quad T = \begin{pmatrix} 0.25 \\ 0.5 \\ 0 \\ 0.25 \\ 0 \end{pmatrix}$$

and $X \in [0, 1]_8^{V \times W}$ is unknown.

Reducts

The context (U, V, R, σ) admits two possible reducts: $Y_1 = \{u_1, u_2, u_3\}$ and $Y_2 = \{u_2, u_3, u_4\}$. Hence, we may consider two different ways of reducing FRE (4) preserving its solution set. According to the reduct $Y_1 = \{u_1, u_2, u_3\}$, it is sufficient preserving the three first equations of FRE (4), whilst the last two equations can be removed.

$$R_{Y_1} \odot_\sigma X = T_{Y_1} \tag{6}$$

which corresponds to the system:

$$\begin{array}{rcrcrcrcrl}
0.75 \,\&_1\, x_1 & \vee & 0.5 \,\&_1\, x_2 & \vee & 0 \,\&_2\, x_3 & \vee & 0.5 \,\&_1\, x_4 & \vee & 0.5 \,\&_2\, x_5 & = 0.25 \\
0.5 \,\&_1\, x_1 & \vee & 0.25 \,\&_1\, x_2 & \vee & 0.25 \,\&_2\, x_3 & \vee & 0.75 \,\&_1\, x_4 & \vee & 1 \,\&_2\, x_5 & = 0.5 \\
0.75 \,\&_1\, x_1 & \vee & 0.5 \,\&_1\, x_2 & \vee & 0.125 \,\&_2\, x_3 & \vee & 0 \,\&_1\, x_4 & \vee & 0.375 \,\&_2\, x_5 & = 0
\end{array}$$

Set of solutions of the reduced FRE

J. C. Díaz-Moreno and J. Medina.
Using concept lattice theory to obtain the set of solutions of multi-adjoint relation equations.
Information Sciences, 266(0):218–225, 2014.

Considering the context $(Y_1, V, R_{Y_1}, \sigma_{|Y_1 \times V})$ we calculate the concept whose intent is

$$T_{Y_1} = \begin{pmatrix} 0.25 \\ 0.5 \\ 0 \end{pmatrix}$$

obtaining $(0, 0, 0, 0.875, 0)$.

The whole set of solutions

The solution set of FRE (5) is given by

$$((0,0,0,0.875,0)] \setminus ((0,0,0,0.625,0)]$$

i.e. $\{X \in [0,1]_8^V \mid X \leq (0,0,0,0.875,0), X \not\leq (0,0,0,0.625,0)\}$. Consequently, we conclude that there are two solutions of the FRE (5), which implies solutions of FRE (4).

$$X_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.875 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.75 \\ 0 \end{pmatrix}$$

Approximating unsolvable FRE

A difficulty that often arises when working with FRE is their lack of solutions, i.e., their unsolvability.

Conservative and optimistic approximations using an associated concept lattice to each FRE

M. E. Cornejo, J. C. Díaz-Moreno, and J. Medina.
Multi-adjoint relation equations: A decision support system for fuzzy logic.
International Journal of Intelligent Systems, 32:778–800, 2017.

Approximation methods that achieve solvability by eliminating redundant information.

D. Lobo, V. López-Marchante, and J. Medina.
Reducing fuzzy relation equations via concept lattices.
Fuzzy Sets and Systems, 463:108465, 2023.

Conclusions and future work

- POCL/OOCL are generalization of RST with the philosophy of FCA.
- We have characterized the solvability of FRE with the POCL/OOCL concepts.
- Multi-adjoint relation equations are the most flexible relation equation that can be solved, at the moment.
- We have presented several mechanisms in order to compute approximate solutions for unsolvable multi-adjoint relation equations.
- As a future work, we will continue exploiting the relationship among FCA and FRE.
- We will apply the obtained results to real-life problems.

THANK YOU FOR YOUR ATTENTION

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IJCRS 2024
May 19th
Cádiz, Spain – Halifax, Canada