Fuzzy orthopartitions as models dealing with vague and uncertain information

Stefania Boffa

(this research is done with Professor Davide Ciucci)

University of Milano-Bicocca, Italy

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Schedule

1. **Fuzzy orthopartitions**;
2. Fuzzy orthopartitions and other models for partitions in presence of partial knowledge (**orthopartitions, Ruspini partitions, and credal partitions**);
3. **Entropy measures and an ordering** on fuzzy orthopartitions;
4. Future and potential developments of fuzzy orthopartitions.

Partitions

A **standard partition** of U is made of $C_1, \ldots, C_n \subseteq U$ such that

$$C_i \cap C_j = \emptyset \quad \text{and} \quad C_1 \cup \ldots \cup C_n = U.$$

$C_1, \ldots, C_n$ are called **equivalence classes**.

Example:

$C_1 = \{a, b\}$ and $C_2 = \{c\}$ form a partition of $\{a, b, c\}$.

In a **standard partition**,
each element **fully** belongs to **exactly one** class.

Fuzzy orthopartitions

Fuzzy orthopartition:
each element belongs to a class **with a degree** x so that

- $x \in [0, 1]$ (**vagueness**);
- $a \leq x \leq b$, with $[a, b] \subseteq [0, 1]$ (**uncertainty**).

Example:

- $a \in C_1$ (standard partition $\{C_1 = \{a, b\}, C_2 = \{c\}\}$);
- $a \in C_1$ with an unknown degree in [0.3,0.7] (fuzzy orthopartition).

Fuzzy orthopartitions are generalized partitions involving both **vagueness** and **uncertainty**.

Fuzzy orthopartitions

Fuzzy orthopartitions generalize

- **standard orthopartitions**
  (partitions with uncertainty),
- **Ruspini partitions**
  (partitions with vagueness).

Orthopartitions based on classical sets (1/2)

An orthopartition is understood as a partition where

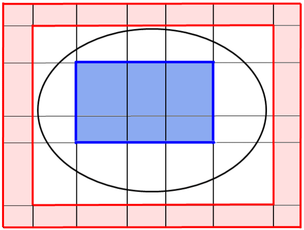
- the **membership class of some elements is known with certainty**,
- whereas **the membership class of the remaining elements is completely unknown**.

An **orthopartition** is a generalized partition where blocks are **orthopairs**, namely **pairs of disjoint subsets**.

Example:
$C_1 = (\{a, b\}, \{c\})$ and $C_2 = (\{c\}, \{a, b\})$ form an orthopartition of $\{a, b, c, d\}$.

$$a, b \in C_1, \quad c \in C_2, \quad d \in \mathbf{?}.$$

Orthopairs and Rough Sets



U : universe
$\mathcal{R}$: equivalence relation
X : subset of U

Orthopair:

$(\mathcal{L}(X), U \setminus \mathcal{U}(X))$

Orthopartitions based on classical sets (2/2)

$O = \{(P_1, N_1), \ldots, (P_n, N_n)\}$ is an **orthopartition** of U if and only if

1. $P_i \cap P_j = \emptyset$, $P_i \cap B_j = \emptyset$, and $P_j \cap B_i = \emptyset$, for each $i \neq j$ (**the classes are disjoint**),
2. $\bigcup_{i=1}^{n}(P_i \cup B_i) = U$ (**covering requirement**),
3. for each $u \in U$, if $u \in U$ then there exists $j \neq i$ such that $u \in B_j$ (**the uncertainty must be shared by at least two classes**),

where $B_i = U \setminus P_i \cup N_i$.

Andrea Campagner and Davide Ciucci. "Orthopartitions and soft clustering: soft mutual information measures for clustering validation." Knowledge-Based Systems.

Ruspini partitions

A **Ruspini partition** is a generalized partition
where equivalence classes are **fuzzy sets**.
Enrique H Ruspini. A new approach to clustering. Information and control, 15(1):22–32, 1969

- **Fuzzy sets** are sets whose elements have degrees of membership;
- A fuzzy set is a function $\pi : U \to [0,1]$, where
  $\pi(u)$ is the degree to which u belongs to U.

A **Ruspini partition** of U is a family $\pi = \{\pi_1, \ldots, \pi_n\}$ of fuzzy sets on U ($\pi_i : U \to [0,1]$) such that

$$\pi_1(u) + \ldots + \pi_n(u) = 1, \text{ for each } u \in U.$$

Example:
$\{\pi_1, \pi_2\}$ **is a Ruspini partition of** $\{a, b, c\}$, where

$$\pi_1(a) + \pi_2(a) = 0.2 + 0.8 = 1, \ \pi_1(b) + \pi_2(b) = 0.5 + 0.5 = 1, \text{ and}$$
$$\pi_1(c) + \pi_2(c) = 1 + 0 = 1.$$

Intuitionistic fuzzy sets

Fuzzy orthopartitions are collections of intuitionistic fuzzy sets.

Definition:
An **intuitionistic fuzzy set** A of a universe U is defined as

$$A = (\mu, \nu)$$

where the maps $\mu : U \to [0, 1]$ and $\nu : U \to [0, 1]$ satisfying the following condition: for each $u \in U$

$$\mu(u) + \nu(u) \leq 1.$$

- $\mu(u)$: **degree of membership**,
- $\nu(u)$: **degree of non-membership of u to A**,
- $h(u) = 1 - (\mu(u) + \nu(u))$: **hesitation margin of u to A**.

Atanassov, Krassimir T., and Krassimir T. Atanassov. Intuitionistic fuzzy sets. Physica-Verlag HD, 1999.

Fuzzy orthopartitions

An element u belongs to the class i with a degree of the interval $[\mu_i(u), \mu_i(u) + h_i(u)]$, where $\mu_i(u) + h_i(u) = 1 - \nu_i(u)$.

$\mathcal{O} = \{(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}$ is a **fuzzy orthopartition** of U if and only if for each $u \in U$:

1. $\sum_{i=1}^{n} \mu_i(u) \leq 1$ (**disjoint blocks**),
2. $\mu_i(u) + h_j(u) \leq 1$, for each $i \neq j$ (**disjoint classes**),
3. $\sum_{i=1}^{n} \mu_i(u) + h_i(u) \geq 1$ (**covering condition**),
4. for each $i \in \{1, \ldots, n\}$ with $h_i(u) > 0$, there exists $j \in \{1, \ldots, n\} \setminus \{i\}$ such that $h_j(u) > 0$ (**the uncertainty cannot regard only one class**).

Stefania Boffa and Davide Ciucci. "Logical entropy and aggregation of fuzzy orthopartitions." Fuzzy Sets and Systems 455 (2023): 77-101.

Example of a fuzzy orthopartition

The intuitionistic fuzzy sets $(\mu_1, \nu_1), (\mu_2, \nu_2)$, and (μ_3, ν_3)

- form a fuzzy orthopartition of $\{u, v, z\}$.

	T_1 (μ_1, ν_1)	T_2 (μ_2, ν_2)	T_3 (μ_3, ν_3)
u	$(0.3, 0.2)$	$(0.4, 0.3)$	$(0, 0.7)$
v	$(0.2, 0.4)$	$(0.3, 0.2)$	$(0.3, 0.3)$
z	$(0, 0.5)$	$(0.3, 0.4)$	$(0.6, 0.2)$

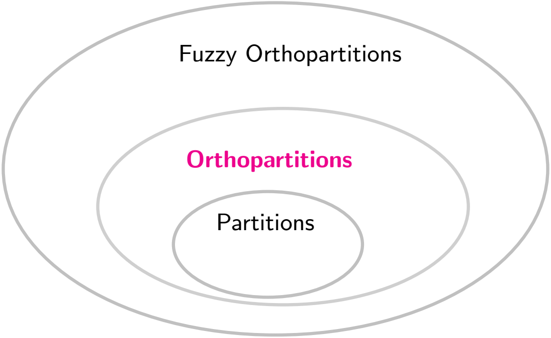
- describe the interest in three topics T_1, T_2, and T_3 of three different groups u, v, and z of users of a social network;

the users of u are interested in the topic T_2
with a degree
from $\mu_2(u) = 0.4$ to $1 - \nu_2(u) = \mu_2(u) + h_2(u) = 0.7$.

Orthopartitions as special fuzzy orthopartitions

An intuitionistic fuzzy set (μ, ν) so that $\mu, \nu : U \to \{0, 1\}$ is an **orthopair** on U.

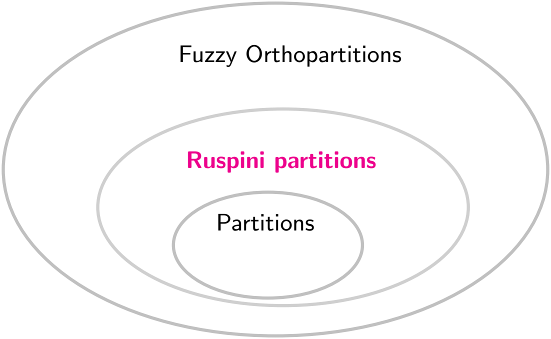
Theorem:
Let $\mathcal{O} = \{(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}$ be a fuzzy orthopartition made of **orthopairs**, then **$\mathcal{O}$ is a crisp orthopartition of U.**



Ruspini partitions as special fuzzy orthopartitions

An intuitionistic fuzzy set (μ, ν) so that $h(u) = 0$ for each $u \in U$ is a **fuzzy set** on U.

Theorem:
Let $\mathcal{O} = \{(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}$ be a fuzzy orthopartition made of **fuzzy sets**, then **$\mathcal{O}$ is a Ruspini partition of U.**



Compatible Ruspini partitions

A fuzzy orthopartition $\mathcal{O}$ corresponds to a class of Ruspini partitions $\Pi_{\mathcal{O}}$.

$\pi \in \Pi_{\mathcal{O}}$ (π is compatible with $\mathcal{O}$) if and only if $\mathcal{O}$ could coincide with π, once the uncertainty is eliminated from $\mathcal{O}$

namely, the membership degree of elements to the classes is precisely known.

Compatible Ruspini partitions

O becames a Rupini partition of Π_O, when we are able to know the **specific value of the interval $[\mu_i(u), \mu_i(u)+h_i(u)]$** representing the degree to which u belongs to the class i.

- Let $\pi = \{\pi_1, \dots, \pi_n\}$ be a Ruspini partition of U.
- Let $\mathcal{O} = \{(\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\}$ be a fuzzy orthopartition of U.

π is **compatible** with $\mathcal{O}$ if and only if $\mu_i \leq \pi_i \leq \mu_i + h_i$.

Example:

- Let $\pi = \{\pi_1, \pi_2\}$ be a Ruspini partition of $\{u\}$.
- Let $\mathcal{O} = \{(\mu_1, \nu_1), (\mu_2, \nu_2)\}$ be a fuzzy orthopartition of $\{u\}$.

$\mu_1(u)$	$\mu_1(u) + h_1(u)$	$\pi_1(u)$
0.2	0.5	0.4

$\mu_2(u)$	$\mu_2(u) + h_2(u)$	$\pi_2(u)$
0.5	0.6	0.6

$\{\pi_1, \pi_2\}$ is compatible with $\mathcal{O}$.

Credal partitions

- Let $U = \{u_1, \ldots, u_l\}$ be a universe;
- let $C = \{C_1, \ldots, C_n\}$ be a standard partition of U.

A **credal partition** is a collection

$$m = \{m_1, \ldots, m_l\}$$

of **basic belief assignments**.

Thierry Denoeux and Marie-Hélène Masson. Evclus: evidential clustering of proximity data. IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics), 34(1):95–109, 2004.

Basic belief assignments

Let $A \subseteq \mathcal{C}$, $m_i(A) \in [0, 1]$, called **mass of belief**, quantifies the evidence supporting the claim

"u_i belongs to a class of A".

Example:

Let $\mathcal{C} = \{C_1, C_2, C_3\}$ be a partition of $\{u_1, u_2, u_3, u_4\}$,
$m = \{m_1, m_2, m_3, m_4\}$ is a credal partition of $\{u_1, u_2, u_3, u_4\}$.

A	$m_1(A)$	$m_2(A)$	$m_3(A)$	$m_4(A)$
$\emptyset$	0.2	0	0	0.1
$\{C_1\}$	0	0.1	0.1	0.1
$\{C_2\}$	0.3	0.3	0	0.1
$\{C_3\}$	0	0.1	0	0.2
$\{C_1, C_2\}$	0.2	0	0.1	0.1
$\{C_1, C_3\}$	0.1	0	0	0.1
$\{C_2, C_3\}$	0	0	0.2	0.3
$\mathcal{C}$	0.2	0.5	0.6	0

Credal partitions and Fuzzy probabilistic partitions

- Credal partitions subsume the concept of **fuzzy probabilistic partitions**.
- $m = \{m_1, \ldots, m_l\}$ is a fuzzy probabilistic partition iff $m_1, \ldots, m_l$ are **Bayesian bbas** ($m_i(A) = 0$ for each $|A| > 1$).

Example:

A	$m_1(A)$	$m_2(A)$	$m_3(A)$
$\emptyset$	0	0	0
C_1	0.1	0.2	0.4
C_2	0.8	0.5	0.4
C_3	0.1	0.3	0.2
$\{C_1, C_2\}$	0	0	0
$\{C_2, C_3\}$	0	0	0
$\{C_1, C_3\}$	0	0	0
$\{C_1, C_2, C_3\}$	0	0	0

Compatible fuzzy probabilistic partitions

A credal partition m corresponds to a class of fuzzy probabilistic partitions denoted with Π_m.

In a dynamic situation, where the **knowledge** about the membership class of the elements

- **is partial** and
- **increases (for example over the time)**

so that partitions become fuzzy probabilistic partitions.

Ruspini and fuzzy probabilistic partitions

Fuzzy probabilistic and Ruspini partitions mathematically coincide. Both assign a degree to each element and class.

Example:

Ruspini partition = Fuzzy orthopartition

	u_1	u_2	u_3
$\pi_1(u)$	0.2	0.5	0.6
$\pi_2(u)$	0.8	0.5	0.4

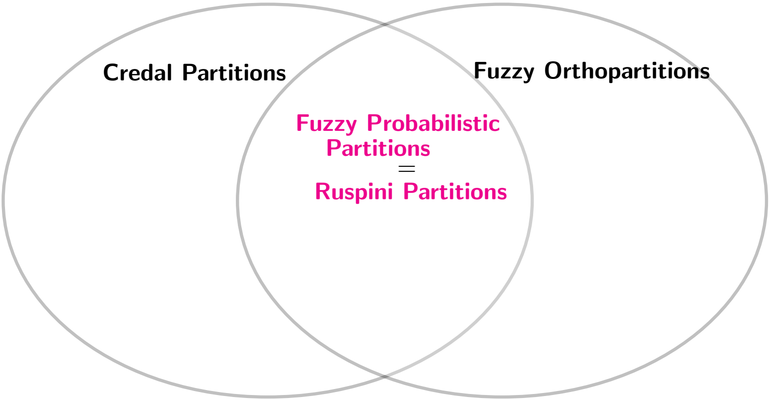
Fuzzy probabilistic partition = Credal partition

A	$m_1(A)$	$m_2(A)$	$m_3(A)$
$\emptyset$	0	0	0
C_1	0.2	0.5	0.6
C_2	0.8	0.5	0.4
C	0	0	0

Credal partitions and Fuzzy orthopartitions

A first correspondence

A fuzzy orthopartition and a credal partition coincide, when they are respectively equal to a Ruspini partition and a fuzzy probabilistic partition that coincide.



Credal partitions and Fuzzy orthopartitions

Second correspondence

- A fuzzy orthopartition O corresponds to a **class of Ruspini partitions Π_O**;
- A credal partition m corresponds to a **class of fuzzy probabilistic partitions Π_m**;
- Ruspini and fuzzy probabilistic partitions coincide.

Theorem:
Let O be a fuzzy orthopartition, then

$$|\{m \text{ credal partition } \mid \Pi_O = \Pi_m\}| = \begin{cases} 0, \\ 1, \\ \infty. \end{cases}$$

Theorem:
Let m be a credal partition, there exists at most a fuzzy orthopartition O so that $\Pi_O = \Pi_m$.

Generalized fuzzy orthopartitions

Definition

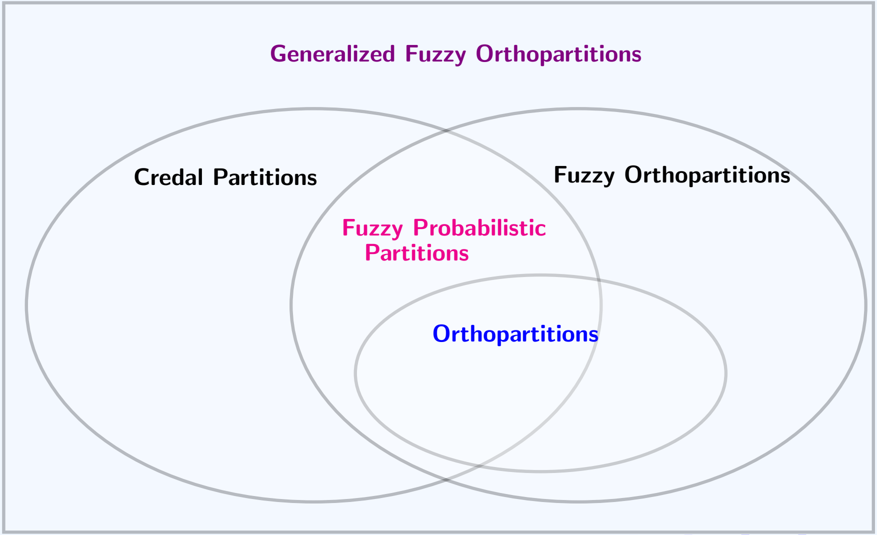
$\mathcal{O} = \{(\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\}$ is a **generalized fuzzy orthopartition** of U if and only if for each $u \in U$:

1. $\sum_{i=1}^{n} \mu_i(u) \leq 1$ (**disjoint blocks**),
2. $\sum_{i=1}^{n} \mu_i(u) + h_i(u) \geq 1$ (**covering condition**).

- Stefania Boffa and Davide Ciucci. Unifying credal partitions and fuzzy orthopartitions. Information Sciences.
- Stefania Boffa and Davide Ciucci. "A correspondence between credal partitions and fuzzy orthopartitions." International Conference on Belief Functions. Cham: Springer International Publishing, 2022.

Generalized fuzzy orthopartitions

Hierarchy



Generalized fuzzy orthopartitions

Interpretation in terms of masses of belief

- Generalized fuzzy orthopartitions: **degrees of membership and non-membership**;
- Credal partitions: **masses of belief**.

Then, we could interpret fuzzy orthopartitions in terms of masses.

A generalized fuzzy orthopartition O can be understood as a credal partition with an additional level of uncertainty: some masses are known and others are not but need to satisfy particular conditions.

O corresponds to one of the credal partitions of $\{m \mid \Pi_O = \Pi_m\}$ once we have enough knowledge so that all masses are determined.

Entropy measures of fuzzy orthopartitions (1/2)

- **Entropy measures for fuzzy orthopartitions**;
- Entropy has an important role in Artificial Intelligence;
- Entropy measures with different scopes, have been defined and studied by taking into account **fuzzy sets, intuitionistic fuzzy sets**, and so on.

Let O be a fuzzy orthopartition, then

- an entropy measure **quantifies the uncertainty** contained in O, **describing the closeness** of O **to a Ruspini partition**.
- the smaller the entropy of O is, the closer O is to a Ruspini partition and less uncertainty contains.

Entropy measures of fuzzy orthopartitions (2/2)

Related articles:

- Stefania Boffa and Davide Ciucci. "Logical entropy and aggregation of fuzzy orthopartitions." Fuzzy Sets and Systems.
- Stefania Boffa, Davide Ciucci, and Christophe Marsala "Extending intuitionistic operations, orderings, and entropy measures on generalized fuzzy orthopartitions" (Submitted)

We introduce entropies following two approaches:

- generalizing the **logic entropy of standard partitions**;
  D. Ellerman, Counting distinctions: on the conceptual foundations of Shannon's information theory, Synthese 168 (2009) 119–149.
- generalizing the **entropies of intuitionistic fuzzy sets;**
  Eulalia Szmidt and Janusz Kacprzyk. New measures of entropy for intuitionistic fuzzy sets. In Ninth Int Conf IFSs Sofia, volume 11, pages 12–20, 2005.

Logical entropy of Ruspini partitions

Definition:
Let $\pi = \{\pi_1, \ldots, \pi_n\}$ be a Ruspini partition of U. Then, the **logical entropy of π** is given by

$$\mathcal{H}(\pi) = \frac{\sum_{(u,v)\in U\times U} dit\pi(u,v)}{|U\times U|}.$$

- $dit\pi(u,v)$ is the **degree of distinction of (u,v)**, and it is interpreted as the capacity of π to distinguish u and v by means of its fuzzy sets $\pi_1, \ldots, \pi_n$.
- Formally,

$$dit\pi(u,v) = \max\{|\pi_i(u) - \pi_i(v)| \text{ s.t. } i \in \{1, \ldots, n\}\}.$$

Proposition:
If π is made of classical sets, then $\mathcal{H}(\pi)$ is the **logical entropy** of a standard partition.

Lower and upper entropy of fuzzy orthopartitions

Let $\mathcal{O}$ be a fuzzy orthopartition of U.

- **Lower entropy of $\mathcal{O}$**: $\mathcal{H}_*(\mathcal{O}) = \min\{\mathcal{H}(\pi) \mid \pi \in \Pi_{\mathcal{O}}\}$,
- **Upper entropy of $\mathcal{O}$**: $\mathcal{H}^*(\mathcal{O}) = \max\{\mathcal{H}(\pi) \mid \pi \in \Pi_{\mathcal{O}}\}$.

They measure the quantity of information contained in $\mathcal{O}$ (the capacity of $\mathcal{O}$ to distinguish elements of U by means of its classes.)

The interval $\mathcal{I}_O = [h_*(O), h^*(O)]$ **is an entropy measure too**.

Example of upper and lower entropies

Suppose that $(\mu_1, \nu_1), (\mu_2, \nu_2)$, and (μ_3, ν_3)

- express the interest in three topics T_1, T_2, and T_3 of three different groups u, v, and z of users of a social network;
- form a fuzzy orthopartition $\mathcal{O}$ of $\{u, v, z\}$.

	μ_1	ν_1	μ_2	ν_2	μ_3	ν_3
u	0.3	0.2	0.4	0.3	0	0.7
v	0.2	0.4	0.3	0.2	0.3	0.3
z	0	0.5	0.3	0.4	0.6	0.2

The lower and upper entropies of $\mathcal{O}$ are

$$\mathcal{H}_*(\mathcal{O}) = 0.13 \quad \text{and} \quad \mathcal{H}^*(\mathcal{O}) = 0.31.$$

They measure how much the interests of u, v, and z diverge (valued w.r.t. $\{T_1, T_2, T_3\}$).

Hence, they diverge with a degree between 0.13 and 0.31.

How to compute the lower and upper entropy

1. The problem to find π_* and π^* can be transformed in a **constrained optimization problem** (finding the maximum and minimum points of a function subject to a pair of constraints).
2. It is converted into a **linear programming problem**.
3. The **optimal solutions** can be computed by using one of the standard techniques in linear programming like the **Simplex method**.

Stefania Boffa and Davide Ciucci. "Logical entropy and aggregation of fuzzy orthopartitions." Fuzzy Sets and Systems 455 (2023): 77-101.

An ordering on fuzzy orthopartitions

Ordering $\leq$ on intuitionistic fuzzy sets

Let $A_1 = (\mu_1, \nu_1)$ and $A_2 = (\mu_2, \nu_2)$ be intuitionistic fuzzy sets.

$$A_1 \leq A_2 \quad \text{iff} \quad \mu_1(u) \leq \mu_2(u) \text{ and } \nu_2(u) \leq \nu_1(u), \;\; \forall\, u \in U.$$

A_1 is **less fuzzy** than A_2, i.e. A_2 **is closer than** A_1 to a fuzzy set because it contains less uncertainty.

Extension of $\leq$ on fuzzy orthopartition

Let O_1 and O_2 be fuzzy orthopartitions.

$$O_1 \preceq O_2 \;\; \text{iff} \;\; (\mu_1, \nu_1)_i \leq (\mu_2, \nu_2)_i, \; \forall\, i \in \{1, \ldots, n\}.$$

If $O_1 \preceq O_2$ then

- O_1 is **less fuzzy** than O_2;
  O_2 is closer than O_1 to a Ruspini partition because it contains less uncertainty.
- O_2 is a **refinement** of O_1;
  in a dynamic situation, we can imagine O_2 as an evolution of O_1 once more information is known about the elements of the universe U.
- **Monotonicity:** $\mathcal{I}(O_1) \subseteq \mathcal{I}(O_2)$;
  ($\mathcal{I}(O_1)$ is a closed subinterval of $\mathcal{I}(O_2)$);
  A fuzzy orthpartition with a lower entropy is closer to a Ruspini partition.

Conclusions and future directions

Fuzzy orthopartitions have been introduced to represent generalized partitions, according to the need of considering classification in presence of uncertainty and vagueness.

In the future, we intend to

- extract fuzzy orthopartitions from data (using the correspondence between fuzzy orthopartitions and credal partitions and existing method to generate credal partitions from data);
- compare the existing measures of uncertainty in the setting of credal and fuzzy orthopartitions;
- compare fuzzy orthopartitions with other generalized partitions (*three-way fuzzy partitions*);
- define fuzzy relations with uncertainty that are equivalent to fuzzy orthopartitions (similarly to the connection between standard partitions and equivalence relations);
- view fuzzy orthopartitions as contingency table by giving a new semantics to intuitionistic fuzzy sets in terms of relative frequencies (work in progress).

Thanks for the attention!